DETERMINISTIC CRAMÉR-RAO BOUND FOR SYMMETRIC PARAFAC MODEL
WITH APPLICATION TO BLIND SPATIAL SIGNATURE ESTIMATION

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ABSTRACT
The symmetric PARAllel FACtor analysis (PARAFAC) model has found numerous applications in array signal processing and communications. In this paper, we derive the deterministic Cramér-Rao Bound (CRB) for the symmetric PARAFAC model and illustrate the obtained results using an example with spatial signature estimation in sensor arrays.

1. INTRODUCTION AND DATA MODEL
A family of blind array processing algorithms including ESPRIT-like method [1]-[2], Second Order Blind Identification (SOBI) algorithm [3]-[4] and blind spatial signature estimation method based on time-varying user power loading [5] exploit the models which essentially share the same structure called a symmetric PARAFAC model.

The CRB analysis for the methods based on the symmetric PARAFAC model is of great interest. In this paper, we derive such CRB in a closed form and illustrate the obtained results using an example with spatial signature estimation in sensor arrays.

Let an array of K sensors receive the signals from M narrow-band sources. The K x 1 snapshot vector of antenna array outputs can be written as

\[ y(n) = As(n) + v(n) \]  \hspace{1cm} (1)

where \( A = [a_1, \ldots, a_M] \) is the K x M complex matrix of the user spatial signatures, \( a_m = [a_{1,m}, \ldots, a_{K,m}]^T \) is the K x 1 complex spatial signature of the mth user, \( s(n) = [s_1(n), \ldots, s_M(n)]^T \) is the M x 1 complex vector of the user waveforms, \( v(n) = [v_1(n), \ldots, v_K(n)]^T \) is the K x 1 vector of additive spatially and temporarily white complex Gaussian noise, and \( (\cdot)^T \) denotes the transpose. Assuming that there is a block of N snapshots available, the model (1) can be written as

\[ Y = AS + V \]  \hspace{1cm} (2)

where \( Y = [y(1), \ldots, y(N)] \) is the K x N array data matrix, \( S = [s(1), \ldots, s(N)] \) is the M x N user waveform matrix, and \( V = [v(1), \ldots, v(N)] \) is the K x N sensor noise matrix.

Assuming that the user signals are uncorrelated with each other and sensor noise, the array covariance matrix of the received signals can be written as

\[ R = \mathbb{E}\{y(n)y^H(n)\} = AQ^H + \sigma^2 I \]  \hspace{1cm} (3)

where \( Q = \mathbb{E}\{s(n)s^H(n)\} \) is the diagonal covariance matrix of the signal waveforms, \( \sigma^2 \) is the sensor noise variance, \( I \) is the identity matrix, and \( (\cdot)^H \) denotes the Hermitian transpose.

2. SYMMETRIC PARAFAC MODEL
Often, it is required to estimate the matrix \( A \) in (2) based on the observations \( Y \) only. In the multiple user case, this is not possible to do with only one known covariance matrix (3) because the matrix \( A \) can be estimated from \( R \) only up to an arbitrary unknown unitary matrix. To provide a unique estimate of \( A \), several covariance matrices have to be used, see [1]-[5].

In this paper, following the approach of [5] with artificial user power loading, we assume that a set of covariance matrices is obtained by dividing uniformly the whole data block of N snapshots into P sub-blocks, each of \( N_c = \lceil \frac{N}{P} \rceil \) snapshots, where \( \lceil x \rceil \) denotes the largest integer less than \( x \). The transmitted power of each user is assumed to be fixed within each particular sub-block while is changed from one sub-block to another. Using such power loading scheme, we obtain that the received snapshots within any \( p \)th sub-block correspond to the following covariance matrix

\[ R(p) = AQ(p)A^H + \sigma^2 I \]  \hspace{1cm} (4)

where \( Q(p) \) is the diagonal covariance matrix of the user waveforms in the \( p \)th sub-block and \( p = 1, \ldots, P \).

In practice, the noise power can be estimated and then subtracted from the covariance matrix (4). Let us stack the P matrices \( R(p) = \sigma^2 I, \quad p = 1, \ldots, P \) together to form a three-way array \( \mathbf{R} \). This three-way array has a symmetry dictated by the symmetry of the matrices \( R(p) = \sigma^2 I \). The \( (i, i, p) \)th element of such an array can be written as

\[ r_{i,i,p} = \{R(p)\}_{i,i} = \sum_{m=1}^{M} a_{i,m}^* \nu_m(p)a_{i,m} \]  \hspace{1cm} (5)

where \( \nu_m(p) = [Q(p)]_{m,m} \) is the power of the \( m \)th user in the \( p \)th sub-block and \( (\cdot)^* \) denotes the complex conjugate. Defining the \( P \times M \) matrix \( \mathbf{P} \) as

\[ P = \begin{bmatrix} \nu_1(1) & \cdots & \nu_M(1) \\ \vdots & \ddots & \vdots \\ \nu_1(P) & \cdots & \nu_M(P) \end{bmatrix} \]  \hspace{1cm} (6)
we have that $Q(p) = D_p\{P\}$ for all $p = 1, \ldots, P$ where $D_p\{\}$ is the operator that makes a diagonal matrix by selecting the $p$th row and putting it on the main diagonal while putting zeros elsewhere.

3. DETERMINISTIC CRAMÉR-RAO BOUND

The model (1) for the $n$th sample of the $p$th sub-block can be rewritten as

$$y(p, n) = \mathbf{A} Q^{1/2}(p) \mathbf{\hat{a}}(n) + \mathbf{v}(n), \quad n = (p - 1)N_p + 1, \ldots, pN_p$$

(7)

where $\mathbf{\hat{a}}(n) = [\mathbf{\hat{a}}_1(n), \ldots, \mathbf{\hat{a}}_M(n)]^T = Q^{-1/2}(p) \mathbf{a}(n)$ is the vector of normalized signal waveforms and the normalization is done so that all waveforms have unit powers.

Hence, the observations in the $p$th sub-block satisfy the following model

$$y(p, n) \sim CN(\mu(p, n), \sigma^2 I)$$

(8)

where

$$\mu(p, n) = \mathbf{A} Q^{1/2}(p) \mathbf{\hat{a}}(n), \quad n = (p - 1)N_p + 1, \ldots, pN_p$$

(9)

The unknown parameters of the model (7) are all the entries of $\mathbf{A}$, the diagonal elements of $Q(p)$ ($p = 1, \ldots, P$) and the noise power $\sigma^2$. Note, however, that the latter parameter is decoupled with the other parameters in the Fisher Information Matrix (FIM) [6]. Therefore, without loss of generality, $\sigma^2$ can be excluded from the vector of unknown parameters.

A delicate point regarding the CRB is the inherent permutation and scale ambiguity. To derive a meaningful CRB, we assume that the first row of $\mathbf{A}$ is normalized to $[1, \ldots, 1]^T$ (this removes the scaling ambiguity), and the first row of $\mathbf{P}$ is known and consists of distinct elements (which resolves the permutation ambiguity). Then, the $(2(K-1)M \times 1$ real vector of the unknown parameters is given by

$$\alpha = [\alpha_2^T, \ldots, \alpha_K^T]^T$$

(10)

where $\alpha_k = [\text{Re}(\mathbf{\hat{a}}_k)^T, \text{Im}(\mathbf{\hat{a}}_k)^T]^T$ and $\mathbf{\hat{a}}_k = [\mathbf{\hat{a}}_{k,1}, \ldots, \mathbf{\hat{a}}_{k,M}]^T$.

The $(P - 1)M \times 1$ vector of nuisance parameters can be expressed as

$$\zeta = [\hat{p}(2), \ldots, \hat{p}(P)]^T$$

(11)

where $\hat{p}(p)$ is the $p$th row of the matrix $\mathbf{P}$.

Using (10) and (11), the $(2(K-1)M + (P - 1)M) \times 1$ real vector of unknown parameters can be written as

$$\theta = [\alpha^T, \zeta^T]^T$$

(12)

Theorem: The $(2(K-1)M + (P - 1)M) \times (2(K-1)M + (P - 1)M)$ FIM is given by

$$\begin{bmatrix}
J_{\alpha_2, \alpha_2} & \cdots & J_{\alpha_2, \hat{p}(P)} \\
0 & \cdots & \vdots \\
J_{\alpha_K, \alpha_K} & \cdots & J_{\alpha_K, \hat{p}(P)} \\
J_{\hat{p}(2), \hat{p}(2)} & \cdots & 0 \\
\vdots & \cdots & \vdots \\
J_{\hat{p}(P), \hat{p}(P)} & \cdots & 0 \\
J_{\hat{p}(P), \hat{p}(P)} & \cdots & 0 \\
\end{bmatrix}$$

(13)

where

$$J_{\alpha_2, \alpha_2} = \cdots = J_{\alpha_K, \alpha_K} = \frac{2}{\sigma^2} \begin{bmatrix}
\text{Re}(\mathbf{H}^T \mathbf{Y}) & \text{Im}(\mathbf{H}^T \mathbf{Y}) \\
\text{Im}(\mathbf{H}^T \mathbf{Y}) & \text{Re}(\mathbf{H}^T \mathbf{Y}) \\
\end{bmatrix}$$

(14)

$$J_{\hat{p}(2), \hat{p}(P)} = \frac{2}{\sigma^2} \text{Re}[(\mathbf{G}(p))^H \mathbf{G}(p)]$$

(15)

$$J_{\alpha_k, \hat{p}(P)} = \frac{2}{\sigma^2} (\mathbf{I}_K \otimes \hat{F}(p)) \mathbf{H}(p)$$

(16)

$$\mathbf{Y} = \begin{bmatrix}
\mathbf{f}_1(1) & \cdots & \mathbf{f}_M(1) \\
\vdots & \cdots & \vdots \\
\mathbf{f}_1(P) & \cdots & \mathbf{f}_M(P) \\
\mathbf{h}_{1,1}(p) & \cdots & \mathbf{h}_{1,M}(p) \\
\vdots & \cdots & \vdots \\
\mathbf{h}_{K,1}(p) & \cdots & \mathbf{h}_{K,M}(p) \\
\end{bmatrix}$$

(17)

$$\mathbf{F}(p) = \begin{bmatrix}
\text{Re}(\mathbf{F}(p)) & \text{Im}(\mathbf{F}(p)) \\
\text{Im}(\mathbf{F}(p)) & \text{Re}(\mathbf{F}(p)) \\
\end{bmatrix}$$

(18)

$$\mathbf{F}(p) = [\mathbf{f}_1(p), \ldots, \mathbf{f}_M(p)]$$

(19)

$$\mathbf{H}(p) = [\mathbf{H}_2(p), \ldots, \mathbf{H}_K(p)]^T$$

(20)

$$\mathbf{H}(p) = [\text{Re}(\mathbf{H}(p)), \text{Im}(\mathbf{H}(p))$$

(21)

$$\mathbf{H}_k(p) = [\mathbf{h}_{k,1}(p), \ldots, \mathbf{h}_{k,M}(p)]$$

(22)

$$\mathbf{f}_m(p) = \sqrt{\mu_m(p)} [\mathbf{a}_m(1) N_p + 1, \ldots, \sqrt{\mu_m(p)} N_p]^T$$

(23)

$$\mathbf{h}_{k,m}(p) = \frac{[\mathbf{a}_{k,m} \mathbf{\hat{a}}_{m}(1) N_p + 1, \ldots, \mathbf{a}_{k,m} \mathbf{\hat{a}}_{m}(N_p)]^T}{2 \sqrt{\mu_m(p)}}$$

(24)

and $\otimes$ denotes the Kronecker matrix product.

The $(K-1)M \times (K-1)M$ spatial signature-related block of the CRB matrix is given in the closed form as

$$\text{CRB}_{\alpha, \alpha} = \frac{2}{\sigma^2} \sum_{p=2}^{P} (\mathbf{I}_K \otimes \hat{F}(p)) \mathbf{H}(p) \times \left[ \text{Re}(\mathbf{G}(p)^H \mathbf{G}(p)) \right]^{-1} \mathbf{H}(p)^T (\mathbf{I}_K \otimes \hat{F}(p))^T$$

(25)

where the upper-left block of the FIM can be expressed as

$$\text{J}_{\alpha, \alpha} = \frac{2}{\sigma^2} \mathbf{I}_K \otimes \left[ \text{Re}(\mathbf{H}^T \mathbf{Y}) \right] \text{Im}(\mathbf{H}^T \mathbf{Y}) \text{Re}(\mathbf{H}^T \mathbf{Y})$$

(26)

Proof: The $(i, k)$th element of the FIM is given by

$$\text{FIM}_{i,k} = \frac{2}{\sigma^2} \sum_{p=1}^{P} \sum_{m=1}^{m_N} \sum_{n=1}^{N_p} \text{Re} \left( \frac{\partial \mu_m(p, n)}{\partial \mu} \frac{\partial \mu(p, n)}{\partial \theta_k} \right)$$

(27)
Using (9) along with (27), we have

$$\frac{\partial \mu(p, n)}{\partial \text{Re}(a_{k,m})} = \sqrt{\nu_m(p)} \hat{a}_m(n) e_k$$  \hspace{2cm} (28)

$$\frac{\partial \mu(p, n)}{\partial \text{Im}(a_{k,m})} = \sqrt{\nu_m(p)} \hat{a}_m(n) e_k$$  \hspace{2cm} (29)

$$\frac{\partial \mu(p, n)}{\partial \nu_m(p)} = \left[ \begin{array}{c} \frac{\alpha_{k,m} e_m(n)}{2 \sqrt{\nu_m(p)}} \\ \vdots \\ \frac{\alpha_{K,m} e_m(n)}{2 \sqrt{\nu_m(p)}} \end{array} \right]$$  \hspace{2cm} (30)

where $e_k$ is the vector containing one in the $k$th position and zeros elsewhere.

Using (28) and (29) along with (27) we obtain that

$$J_{\text{Re}(a_{k,m})} = J_{\text{Im}(a_{k,m})} = \frac{2}{\sigma^2} \sum_{p=1}^{P} \sum_{n=p}^{N_p} \text{Re} \left\{ \sqrt{\nu_m(p)} e_m(n) \hat{a}_m(n) \right\}$$  \hspace{2cm} (31)

where $\xi_m = [f_k^H(1), \ldots, f_k^H(P)]^T$.

Similarly,

$$J_{\text{Im}(a_{k,m})} = J_{\text{Re}(a_{k,m})} = \frac{2}{\sigma^2} \text{Re} \left\{ \xi_m^H \right\}$$  \hspace{2cm} (32)

Therefore,

$$J_{\text{Re}(a_k), \text{Re}(a_k)} = J_{\text{Im}(a_k), \text{Im}(a_k)} = \frac{2}{\sigma^2} \left[ \begin{array}{cc} \text{Re} \left\{ \xi_k^H \xi_k \right\} & \ldots \\ \ldots & \ldots \end{array} \right]$$  \hspace{2cm} (33)

and

$$J_{\text{Im}(a_k), \text{Re}(a_k)} = J_{\text{Re}(a_k), \text{Im}(a_k)} = \frac{2}{\sigma^2} \left[ \begin{array}{cc} \text{Im} \left\{ \xi_k^H \xi_k \right\} & \ldots \\ \ldots & \ldots \end{array} \right]$$  \hspace{2cm} (34)

Using (33) and (34), we obtain (13). Note that the right-hand side of (13) does not depend on the index $k$. Hence,

$$J_{\alpha, \alpha} = \left[ \begin{array}{cc} J_{\sigma_1, \sigma_2} & 0 \\ 0 & J_{\alpha_K, \alpha_K} \end{array} \right]$$  \hspace{2cm} (35)

Next, using (30) along with (27) we can write for $p = 2, \ldots, P$ and $m, l = 1, \ldots, M$

$$[J_{\rho(p), \rho(p)}]_{m,l} = \frac{2}{\sigma^2} \left[ \begin{array}{c} \sum_{n=\text{p}(p)}^{N_p} \text{Re} \left\{ \frac{\alpha_{k,m} \hat{a}_m(n) e_k}{2 \sqrt{\nu_m(p)}} \frac{\alpha_{k,l} \hat{a}_l(n) e_k}{2 \sqrt{\nu_l(p)}} \right\} \\ \cdots \\ \sum_{n=\text{p}(p)}^{N_p} \text{Re} \left\{ \frac{\alpha_{k,m} \hat{a}_m(n) e_k}{2 \sqrt{\nu_m(p)}} \frac{\alpha_{k,l} \hat{a}_l(n) e_k}{2 \sqrt{\nu_l(p)}} \right\} \end{array} \right]$$  \hspace{2cm} (36)

where $c_m(p) = [h_1^2(m(p)), \ldots, h_K^2(m(p))]^T$. Stacking all $M^2$ elements given by (36) in one matrix we have for $p = 2, \ldots, P$

$$J_{\rho(p), \rho(p)} = \frac{2}{\sigma^2} \left[ \begin{array}{c} \text{Re} \left\{ c_1^H(p) c_1(p) \right\} \\ \cdots \\ \text{Re} \left\{ c_M^H(p) c_M(p) \right\} \end{array} \right]$$  \hspace{2cm} (37)

Finally, using (28), (29), and (30) along with (27) we can write for $p = 2, \ldots, P$; $k = 2, \ldots, K$, and $m, l = 1, \ldots, M$

$$[J_{\text{Re}(a_k), \rho(p)}]_{m,l} = \frac{2}{\sigma^2} \left[ \begin{array}{c} \sum_{n=\text{p}(p)}^{N_p} \text{Re} \left\{ \frac{1}{2 \sqrt{\nu_m(p)}} \hat{a}_m(n) e_k \hat{a}_k(n) \right\} \\ \cdots \\ \sum_{n=\text{p}(p)}^{N_p} \text{Re} \left\{ \frac{1}{2 \sqrt{\nu_l(p)}} \hat{a}_l(n) e_k \hat{a}_k(n) \right\} \end{array} \right]$$  \hspace{2cm} (38)

and

$$[J_{\text{Im}(a_k), \rho(p)}]_{m,l} = \frac{2}{\sigma^2} \left[ \begin{array}{c} \sum_{n=\text{p}(p)}^{N_p} \text{Re} \left\{ \frac{1}{2 \sqrt{\nu_m(p)}} \hat{a}_m(n) e_k \hat{a}_k(n) \right\} \\ \cdots \\ \sum_{n=\text{p}(p)}^{N_p} \text{Re} \left\{ \frac{1}{2 \sqrt{\nu_l(p)}} \hat{a}_l(n) e_k \hat{a}_k(n) \right\} \end{array} \right]$$  \hspace{2cm} (39)

Collecting all $(K - 1)M^2$ elements given by (38) and $(K - 1)M^2$ elements given by (39) in one matrix, we obtain for $p = 2, \ldots, P$

$$J_{\alpha, \rho(p)} = \frac{2}{\sigma^2} \left[ \begin{array}{c} \text{Re} \left\{ F^H(p) H_{\alpha}(p) \right\} \\ \text{Im} \left\{ F^H(p) H_{\alpha}(p) \right\} \end{array} \right]$$  \hspace{2cm} (40)

Observing that

$$\left[ \begin{array}{c} \text{Re} \left\{ F^H(p) H_{\alpha}(p) \right\} \\ \text{Im} \left\{ F^H(p) H_{\alpha}(p) \right\} \end{array} \right] = \tilde{F}(p) \tilde{H}(p)$$  \hspace{2cm} (41)

we can further simplify (40) to

$$J_{\alpha, \rho(p)} = \frac{2}{\sigma^2} \left( I_K \otimes \tilde{F}(p) \right) \tilde{H}(p)$$  \hspace{2cm} (42)

Also, note that

$$J_{\alpha, \rho(p)} = J_{\rho(p), \alpha}$$  \hspace{2cm} (43)
Using (35), (37), (42) and (43) we obtain the expressions (13)-(24).

Computing the CRB for $\theta$ requires the inverse of the $(2(K + 1)M + (P + 1)M) \times (2(K + 1)M + (P + 1)M)$ FIM matrix. Our objective is to obtain the CRB associated with the vector parameter $\alpha$ only, avoiding the inverse of the full FIM matrix. Exploiting the fact that the lower-right sub-block of the FIM is a block-diagonal matrix and using the partitioned matrix inversion lemma (see [5], p. 572), after some algebra we obtain (25)-(26) and the proof is complete.

4. SIMULATIONS

In order to test the derived CRB we consider a simple example with spatial signature estimation of a single user and assume that the BPSK signal impinges on the linear array of 4 sensors and unknown geometry from $\theta = 50^\circ$ relative to the broadside direction. It is well known that in the single-user case, a single covariance matrix is sufficient to guarantee the uniqueness of the spatial signature estimate which is given by the principal eigenvector of the sample covariance matrix $\mathbf{R}$.

We compare the Root-Mean-Square Error (RMSE) performance of such a principal eigenvector-based estimator with the derived CRB. The RMSE is computed as

$$\text{RMSE} = \sqrt{\frac{1}{L K} \sum_{l=1}^{L} \left\| \hat{\mathbf{a}}(l) - \mathbf{a} \right\|^2},$$

where $L = 100$ is the number of independent simulation runs and $\hat{\mathbf{a}}(l)$ is the estimate of $\mathbf{a}$ obtained in the $l$th run. Note that the scaling ambiguity is eliminated by normalizing $\hat{\mathbf{a}}(l)$ with respect to the first (reference) sensor. The CRB is computed as

$$\text{CRB} = \sqrt{\frac{1}{K} \text{Tr} \left[ \text{CRB}_{\alpha, \alpha} \right]}.$$

Figure 1 displays the RMSE and the CRB versus the number of snapshots $N$ for the Signal-to-Noise Ratio (SNR) equal to 10 dB. Figure 2 shows the same quantities versus the SNR for $N = 100$.

It can be seen that the principal eigenvector-based spatial signature estimator approaches CRB at high SNR. This validates our CRB analysis.

5. CONCLUSIONS

The closed-form expressions for the deterministic CRB for the symmetric PARAFAC model have been derived. The simulation example with blind spatial signature estimation illustrates and validates our CRB analysis.

6. REFERENCES


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